Problem Set 3 – Relativity (GOI36A)

1 Actions, Part 2: Electric Boogaloo

1.) Using the definition of the field strength, the left-hand side of the Bianchi identity is

$$\begin{aligned}\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} &= \partial_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \partial_\nu(\partial_\rho A_\mu - \partial_\mu A_\rho) + \partial_\rho(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= (\partial_\mu\partial_\nu A_\rho - \partial_\nu\partial_\mu A_\rho) + (\partial_\nu\partial_\rho A_\mu - \partial_\rho\partial_\nu A_\mu) + (\partial_\rho\partial_\mu A_\nu - \partial_\mu\partial_\rho A_\nu) \ .\end{aligned} \tag{1.1}$$

In Minkowski spacetime, derivatives commute (i.e. it doesn't matter what order we take derivatives in), and so each of the three terms in parentheses becomes zero. This proves the Bianchi identity.

2.) Consider the case where $\mu = 0$ and $\nu = 0$, while ρ is left free. The left-hand side of the Bianchi identity becomes

$$\partial_0 F_{0\rho} + \partial_0 F_{\rho 0} + \partial_\rho F_{00} \ . \tag{1.2}$$

Now, recall the fact that $F_{\mu\nu}$ is antisymmetric, by definition. This means that $F_{00} = 0$, and $F_{0\rho} = -F_{\rho 0}$. The left-hand side of the Bianchi identity simplifies to

$$\partial_0 F_{0\rho} - \partial_0 F_{0\rho} \ , \tag{1.3}$$

which is trivially zero. So, when $\mu = \nu = 0$, the Bianchi identity is automatic and does not tell us anything interesting about the vector field A_μ. This argument generalizes very easily to show that the Bianchi identity is trivial when $\mu = \rho$, or when $\nu = \rho$.

The Bianchi equations contain **four** non-trivial equations. The easiest way to see this is to note that it doesn't matter what order we pick the three indices in; the Bianchi identity for $\mu = 0$, $\nu = 1$, and $\rho = 2$ are exactly the same as when $\mu = 1$, $\nu = 2$, and $\rho = 0$, etc. So, the only thing that matters is which of the four spacetime indices we do not pick. There are four such options, leading to four independent equations.

Another way to count this is as follows. μ, ν, and ρ must all be different. So, there are four choices for μ, and then from there three remaining choices for ν, and then two left for ρ. Naively, this is $4 \times 3 \times 2 = 24$ equations. However, the Bianchi identity is permutation symmetric; we can permute the indices μ, ν, and ρ and the equation is the same. This cuts down on the number of equations by a factor of three, leaving us with eight. Then, by recalling that $F_{\mu\nu}$ is antisymmetric, we note that the Bianchi identities are invariant under swapping any two indices. This cuts the remaining independent equations down by a factor of two. We thus end up with four.

3.) This generalization is the most natural one, because it just involves replacing ϕ with the vector field A_ν. So, in some sense, you should think of it just as four separate equations of motion, one for each component of A_ν. It is absolutely crucial that we kept the free index ν on A_ν and the dummy index μ that appears in the derivatives distinct, otherwise the Euler-Lagrange equations would start to violate our index notation rules.

4.) Making sure not to repeat an index more than twice, we will write the Euler-Lagrange equations for A_μ as

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} - \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} \right) = 0 \ . \tag{1.4}$$

When we derive the Euler-Lagrange equations, we have to treat A_ν and $\partial_\mu A_\nu$ as independent objects. We also treat J^μ as a constant. So, this tells us that

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} = 4\pi J^\mu \frac{\partial A_\mu}{\partial A_\sigma} = 4\pi J^\mu \delta^\sigma_\mu = 4\pi J^\sigma \ . \tag{1.5}$$

Next, we note that

$$\frac{\partial F_{\mu\nu}}{\partial(\partial_\rho A_\sigma)} = \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\rho A_\sigma)} - \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\rho A_\sigma)} = \delta^\rho_\mu \delta^\sigma_\nu - \delta^\rho_\nu \delta^\sigma_\mu \ . \tag{1.6}$$

By raising the indices on both sides of the above expression with the metric tensor, we also immediately find that

$$\frac{\partial F^{\mu\nu}}{\partial(\partial_\rho A_\sigma)} = \frac{\partial(\partial^\mu A^\nu)}{\partial(\partial_\rho A_\sigma)} - \frac{\partial(\partial^\nu A^\mu)}{\partial(\partial_\rho A_\sigma)} = \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma} \ . \tag{1.7}$$

Using the product rule and applying these expressions, we find that

$$\begin{aligned} \frac{\partial(F_{\mu\nu}F^{\mu\nu})}{\partial(\partial_\rho A_\sigma)} &= F_{\mu\nu}\frac{\partial F^{\mu\nu}}{\partial(\partial_\rho A_\sigma)} + F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial(\partial_\rho A_\sigma)} \\ &= F_{\mu\nu}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma}) + F^{\mu\nu}(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu) \\ &= (F^{\rho\sigma} - F^{\sigma\rho}) + (F^{\rho\sigma} - F^{\sigma\rho}) \\ &= 4F^{\rho\sigma} \ , \end{aligned} \tag{1.8}$$

where in the last line we used antisymmetry of the field strength. Therefore, we find that

$$\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} \right) = -\frac{1}{4} \times 4\partial_\rho F^{\rho\sigma} = -\partial_\rho F^{\rho\sigma} \ . \tag{1.9}$$

Putting (1.5) and (1.9) together, the equation of motion for A_μ is given by

$$4\pi J^\sigma + \partial_\rho F^{\rho\sigma} = 0 \ . \tag{1.10}$$

If we relabel the dummy indices and use ν as the free index instead of μ, we find exactly the the result given in the homework.

Note also that I've computed the Euler-Lagrange equations the long, brute force way here. We can take a shortcut and skip many steps by applying the product rule, as discussed in the note on product rules on Toledo. You can also use antisymmetry of the field strength to cut down on the number of steps, too, and you'll find the same result at the end of the day.

5.) Let $\nu = 0$ in the equations of motion and use $J^0 = \rho$:

$$\partial_\mu F^{\mu 0} = -4\pi J^0 = -4\pi\rho \ . \tag{1.11}$$

Then, by looking at $F^{\mu\nu}$ as given, we can see that

$$\begin{aligned} \partial_\mu F^{\mu 0} &= \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} \\ &= -(\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3) \\ &= -\vec{\nabla} \cdot \vec{E} \ . \end{aligned} \tag{1.12}$$

So, the $\nu = 0$ component of the equations of motion is simply $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$, which is the first of Maxwell's equations.

Next, let $\nu = 1$. The left-hand side of the equation of motion is

$$\begin{aligned} \partial_\mu F^{\mu 1} &= \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} \\ &= \partial_t E_1 - \partial_2 B_3 + \partial_3 B_2 \\ &= \partial_t (\vec{E})_1 - (\vec{\nabla} \times \vec{B})_1 \ . \end{aligned} \tag{1.13}$$

Note that we are using $c = 1$ in this problem so $\partial_0 = \partial_t$. Additionally, we had to recall the definition of the cross product in order to go from the second line to the third line above. The equation of motion for $\nu = 1$ is simply

$$\partial_t (\vec{E})_1 - (\vec{\nabla} \times \vec{B})_1 = -4\pi (\vec{J})_1 \ . \tag{1.14}$$

You should be able to see that we will get precisely the same kind of equation when $\nu = 2$ and $\nu = 3$. We therefore reproduce all three components of the second line of Maxwell's equations.

For the Bianchi identities, let's first choose $\mu = 1$, $\nu = 2$, and $\rho = 3$, i.e. the only index not picked is 0. So,

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0 \ . \tag{1.15}$$

Then, use the metric tensor to show that $F_{23} = \eta_{22}\eta_{33} F^{23} = F^{23} = B_1$. Similarly, $F_{31} = F^{31} = B_2$ and $F_{12} = F^{12} = B_3$. So, the Bianchi identity becomes

$$\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3 = \vec{\nabla} \cdot \vec{B} = 0 \ . \tag{1.16}$$

This is the third line of Maxwell's equations.

Lastly, choose $\mu = 0$, $\nu = 2$, and $\rho = 3$, i.e. the only index not picked is 1:

$$\partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = 0 \ . \tag{1.17}$$

Again, use the metric tensor to express the field strength with lower indices in terms of the field strength with upper indices. You should easily find that $F_{23} = F^{23} = B_1$, $F_{30} = -F^{30} = E_3$, and $F_{02} = -F^{02} = -E_2$. So, the Bianchi identity becomes

$$\begin{aligned} 0 &= \partial_0 B_1 + \partial_2 E_3 - \partial_3 E_2 \\ &= \partial_t (\vec{B})_1 + (\vec{\nabla} \times \vec{E})_1 \ , \end{aligned} \tag{1.18}$$

where in the last line we again had to recall how the cross-product is defined. This is precisely the x-component of the last line of Maxwell's equations. If we repeat the exact same computation but with 2 and 3 as the unchosen indices, we will get the remaining components.

Thus, we have recovered all of Maxwell's equations. The last step is arguing that the explicit forms of $F^{\mu\nu}$ and J^μ given on the problem set are the most general forms possible; if they weren't, then we might be missing something. I claim that they are the most general forms possible, though. To see this, we first count that there are six independent quantities (E_1, E_2, E_3, B_1, B_2, and B_3) in the field strength $F^{\mu\nu}$ and four independent quantities (ρ, J_1, J_2, and J_3) in the source J^μ. Now, $F^{\mu\nu}$ naïvely has sixteen components, but antisymmetry means that the four terms on the diagonal must all vanish, and then the remaining twelve components are related by antisymmetry, leaving us with $(16-4)/2 = 6$ independent components. And, J^μ is a four-component vector with no constraints, and so it has four independent components. Therefore the number of independent components of $F^{\mu\nu}$ and J^μ match exactly with the number of independent quantities we have used to specify them. Therefore the given forms of these tensors are fully general.

6.) Before we proceed, let's first recall that $A_\mu B^\mu = A^\mu B_\mu$ for any vectors A and B. A similar result is true for arbitrary tensors, i.e.

$$S_{\mu_1\cdots\mu_n} T^{\mu_1\cdots\mu_n} = S^{\mu_1\cdots\mu_n} T_{\mu_1\cdots\mu_n} \ . \tag{1.19}$$

In other words, for any dummy index we can always swap which one is raised and which one is lowered. An immediate use of this fact is as follows:

$$\begin{aligned} \partial_\mu (F_{\rho\sigma} F^{\rho\sigma}) &= (\partial_\mu F_{\rho\sigma}) F^{\rho\sigma} + F_{\rho\sigma}(\partial_\mu F^{\rho\sigma}) \\ &= (\partial_\mu F^{\rho\sigma}) F_{\rho\sigma} + F_{\rho\sigma}(\partial_\mu F^{\rho\sigma}) \\ &= 2F_{\rho\sigma}\partial_\mu F^{\rho\sigma} \ . \end{aligned} \tag{1.20}$$

Let's keep these results in mind in the following computation:

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \partial_\mu \left(F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \\ &= (\partial_\mu F^\mu{}_\rho) F^{\nu\rho} + F^\mu{}_\rho \partial_\mu F^{\nu\rho} - \frac{1}{2}\eta^{\mu\nu} F_{\rho\sigma}\partial_\mu F^{\rho\sigma} \\ &= (\partial_\mu F^{\mu\rho}) F^\nu{}_\rho + F_{\mu\rho}\partial^\mu F^{\nu\rho} - \frac{1}{2} F_{\rho\sigma}\partial^\nu F^{\rho\sigma} \\ &= (\partial_\mu F^{\mu\rho}) F^\nu{}_\rho + F_{\mu\rho}\partial^\mu F^{\nu\rho} - \frac{1}{2} F_{\rho\mu}\partial^\nu F^{\rho\mu} \ . \end{aligned} \tag{1.21}$$

In the last line we relabeled dummy indices by letting $\sigma \to \mu$. We will use the results from problem 3 to simplify the first term, and we will use the Bianchi identities on the third terms in order to try to show that the remaining terms cancel:

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= -4\pi J^\rho F^\nu{}_\rho + F_{\mu\rho}\partial^\mu F^{\nu\rho} - \frac{1}{2} F_{\rho\mu} \left(-\partial^\rho F^{\mu\nu} - \partial^\mu F^{\nu\rho} \right) \\ &= -4\pi J_\rho F^{\nu\rho} + F_{\mu\rho}\partial^\mu F^{\nu\rho} + \frac{1}{2} F_{\rho\mu}\partial^\rho F^{\mu\nu} + \frac{1}{2} F_{\rho\mu}\partial^\mu F^{\nu\rho} \ . \end{aligned} \tag{1.22}$$

To see some cancellations, we need to do more index relabelling. In particular, we want to relabel all the dummy indices such that the derivatives all have the same index. So, in the third term, let $\rho \leftrightarrow \mu$:

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= -4\pi J_\rho F^{\nu\rho} + F_{\mu\rho}\partial^\mu F^{\nu\rho} + \frac{1}{2} F_{\mu\rho}\partial^\mu F^{\rho\nu} + \frac{1}{2} F_{\rho\mu}\partial^\mu F^{\nu\rho} \\ &= 4\pi J_\rho F^{\rho\nu} + F_{\mu\rho}\partial^\mu F^{\nu\rho} - \frac{1}{2} F_{\mu\rho}\partial^\mu F^{\nu\rho} - \frac{1}{2} F_{\mu\rho}\partial^\mu F^{\nu\rho} \\ &= 4\pi J_\mu F^{\mu\nu} \ , \end{aligned} \tag{1.23}$$

where we used antisymmetry of $F_{\mu\nu}$ to go from the first to the second line, and in the last we relabelled $\rho \to \mu$. This is what we set out to prove.

2 Spacelike, Timelike, and Null Vectors

7.) T is timelike, so $T_\mu T^\mu < 0$. In components, we will let $T^\mu = (t^0, \vec{t})$. The time-like condition, using this componetn form and the explicit Minkowski metric, means that

$-(t^0)^2 + \vec{t}^2 < 0$, and so $(t^0)^2 > \vec{t} \cdot \vec{t}$. Additionally, we are given that X and T are orthogonal, and so $X_\mu T^\mu = 0$. Writing $X^\mu = (x^0, \vec{x})$, this orthogonality condition means that $-x^0 t^0 + \vec{x} \cdot \vec{t} = 0$, and so $x^0 t^0 = \vec{x} \cdot \vec{t}$.

Our goal is to prove that X is spacelike, i.e. that $X_\mu X^\mu > 0$, which in turn means that $(x^0)^2 < \vec{x} \cdot \vec{x}$. The first step in proving this will be to multiply and divide $(x^0)^2$ by $(t^0)^2$ and then use the orthogonality condition:

$$(x^0)^2 = \frac{(x^0 t^0)^2}{(t^0)^2} = \frac{(\vec{x} \cdot \vec{t})^2}{(t^0)^2} \ . \tag{2.1}$$

Next, the Schwarz inequality tells us that $(\vec{x} \cdot \vec{t})^2 \leq (\vec{x} \cdot \vec{x})(\vec{t} \cdot \vec{t})$, and so

$$(x^0)^2 \leq \frac{(\vec{x} \cdot \vec{x})(\vec{t} \cdot \vec{t})}{(t^0)^2} \ . \tag{2.2}$$

Finally, T is time-like and so $(t^0)^2 > \vec{t} \cdot \vec{t}$, which is equivalent to saying that $\frac{(\vec{t} \cdot \vec{t})}{(t^0)^2} < 1$. Thus, we are finally left with

$$(x^0)^2 \leq (\vec{x} \cdot \vec{x}) \frac{(\vec{t} \cdot \vec{t})}{(t^0)^2} < (\vec{x} \cdot \vec{x}) \ , \tag{2.3}$$

which is exactly the space-like condition on X. Note that in order to derive this we had to divide by t^0 a lot in the equations above. This is okay, because the fact that T is timelike guarantees that $(t^0)^2 > \vec{t} \cdot \vec{t} \geq 0$, and so t^0 must be non-zero.

8.) It is very straightforward to compute that, in this frame,

$$S_\mu T^\mu = \eta_{\mu\nu} S^\mu T^\nu = -S^0 T^0 + S^1 T^1 + S^2 T^2 + S^3 T^3 = 0 \ , \tag{2.4}$$

since $S^0 = 0$ and $T^\mu = 0$ for any $\mu > 0$. Thus, S and T are orthogonal in this inertial frame. To prove that this is true under all inertial frames, we simply have to note that S^μ and T^μ are both tensors, but $S_\mu T^\mu$ has no free indices and is thus a scalar. Scalars do not transform under Lorentz transformations. So, we can do arbitrary Lorentz transformations to go to any other inertial frame, and we will still have $S_\mu T^\mu = 0$. Therefore S and T are orthogonal in all inertial frames.

9.) **Bonus:** Let's first start in an arbitrary inertial frame and write $T^\mu = (t^0, \vec{t})$. Rotations are a perfectly good Lorentz transformation, and we can always do a rotation to go into a frame where $\vec{t}$ lies entirely along the x^1 axis. So, we can always go to a frame in which

$$T^\mu = (t^0, t^1, 0, 0) \ . \tag{2.5}$$

Remember that T is time-like, and so $(t^0)^2 > (t^1)^2$.

Now, we want to prove that there exists a Lorentz transformation that takes us to a frame where $T^{\mu'}$ has only a $\mu' = 0$ component. To do so, consider the following Lorentz transformation:

$$\Lambda^\mu{}_\nu = \frac{1}{\sqrt{(t^0)^2 - (t^1)^2}} \begin{pmatrix} t^0 & -t^1 & 0 & 0 \\ -t^1 & t^0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{2.6}$$

You should check that this satisfies $\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$, as is required for such a transformation to preserve the metric. This Lorentz transformation transforms the vector T as follows:

$$T^{\mu} \to T^{\mu'} = \Lambda^{\mu'}{}_{\mu} T^{\mu} = (\sqrt{(t^0)^2 - (t^1)^2}, 0, 0, 0) \ . \tag{2.7}$$

That is, it boosts us into a frame where T has no spatial components. Note that it is incredibly important that T is time-like; this guarantees that $(t^0)^2 - (t^1)^2 > 0$, and so $\sqrt{(t^0)^2 - (t^1)^2}$ is real and positive. This means that T remains time-like, even after the boost, and is still real. So, this is a perfectly good Lorentz transformation that takes us into a frame where only T^0 is non-zero.

Now, the rest of the proof is easy. S and T are orthogonal, but there is a frame where only T^0 is non-zero. So, we find that

$$0 = S_{\mu}T^{\mu} = -S^0T^0 + S^1T^1 + S^2T^2 + S^3T^3 = -S^0T^0 \ . \tag{2.8}$$

Therefore we must have $S^0 = 0$, and so S has only spatial components in this frame. Thus, there exists a frame where S has only spatial components and T has only the T^0 component, and we are done.

10.) We will write $U^{\mu} = (u^0, \vec{u})$ and $V^{\mu} = (v^0, \vec{v})$. Since U and V are both null vectors, this means that $U_{\mu}U^{\mu} = V_{\mu}V^{\mu} = 0$. Writing these out explicitly in components, we find that $-(u^0)^2 + \vec{u}\cdot\vec{u} = -(v^0)^2 + \vec{v}\cdot\vec{v} = 0$, and thus

$$(u^0)^2 = \vec{u}\cdot\vec{u} \ , \quad (v^0)^2 = \vec{v}\cdot\vec{v} \ . \tag{2.9}$$

Additionally, if U and V are orthogonal, then a similar calculation tells us that

$$u^0v^0 = \vec{u}\cdot\vec{v} \ . \tag{2.10}$$

Comparing (2.9) and (2.10), we immediately can see that

$$(\vec{u}\cdot\vec{v})^2 = (\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) \ . \tag{2.11}$$

This can only be possible if $\vec{u}$ and $\vec{v}$ are parallel vectors. Remember, the dot product can also be written as $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$, where θ is the angle between the vectors. Clearly, we must set $\theta = 0$ (i.e. no angle of separation) in order to satisfy the above equation.

So, since $\vec{u}$ and $\vec{v}$ are parallel, we can write $\vec{v} = \lambda\vec{u}$ for some constant λ. Then, equation (2.10) becomes

$$u^0v^0 = \vec{u}\cdot\vec{v} = \lambda\vec{u}\cdot\vec{u} = \lambda(u^0)^2 \ , \tag{2.12}$$

where in the last step we used (2.9). This clearly tells us that $v^0 = \lambda u^0$. We therefore have shown

$$v^0 = \lambda u^0 \ , \quad \vec{v} = \lambda\vec{u} \ , \tag{2.13}$$

for some constant λ. Since each component of V^{μ} is proportional to each component of U^{μ} with the exact same constant of proportionality, we conclude that V is proportional to U.

3 The Minkowski Metric in Other Coordinates

10.) First, consider $x = r\cos\phi$. If we now consider some infinitesimal variation dx of the coordinate x, the chain rule tells us that it is related to small variations dr and $d\phi$ by

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \phi}d\phi = \cos\phi\, dr - r\sin\phi\, d\phi \ . \tag{3.1}$$

Similarly, the chain rule can be applied to the coordinate relation $y = r\sin\phi$ to show that

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \phi}d\phi = \sin\phi\, dr + r\cos\phi\, d\phi \ . \tag{3.2}$$

The square of these is

$$\begin{aligned} dx^2 &= \cos^2\phi dr^2 - 2r\sin\phi\, \cos\phi\, dr\, d\phi + r^2\sin^2\phi\, d\phi^2 \ , \\ dy^2 &= \sin^2\phi dr^2 + 2r\sin\phi\, \cos\phi\, dr\, d\phi + r^2\cos^2\phi\, d\phi^2 \ . \end{aligned} \tag{3.3}$$

Adding these and making heavy use of $\cos^2\phi + \sin^2\phi = 1$, we find that

$$dx^2 + dy^2 = dr^2 + r^2\, d\phi^2 \ . \tag{3.4}$$

The full Minkowski metric in polar coordinates is thus

$$ds^2 = -c^2dt^2 + dr^2 + r^2\, d\phi^2 + dz^2 \ . \tag{3.5}$$

11.) This problem goes in exactly the same way as the previous problem does. First, consider $x = r\sin\theta\, \cos\phi$ and use the chain rule to conclude

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi \\ &= \sin\theta\, \cos\phi\, dr + r\cos\theta\, \cos\phi\, d\theta - r\sin\theta\, \sin\phi\, d\phi \ . \end{aligned} \tag{3.6}$$

Similarly, we find for dy and dz that

$$\begin{aligned} dy &= \sin\theta\, \sin\phi\, dr + r\cos\theta\, \sin\phi\, d\theta + r\sin\theta\, \cos\phi\, d\phi \ , \\ dz &= \cos\theta\, dr - r\sin\theta\, d\theta \ . \end{aligned} \tag{3.7}$$

Note that there is no ϕ-dependence in z, and so no $d\phi$ term shows up in dz. From here, we simply have to run through the tedious exercise of squaring all three expressions and adding them. The result, after using trig identities to simplify, is

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2 \ . \tag{3.8}$$

The full Minkowski metric in spherical coordinates is hence given by

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2 \ . \tag{3.9}$$